Engineering Notes

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Dynamics of Flexible Hybrid Structures

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Introduction

FLEXIBLE hybrid structures are defined as interconnected rigid and elastic bodies. Such hybrid models have many technical applications, for instance in rotor dynamics, antenna constructions, and satellite dynamics. In the present Note, a formal method is described to obtain the equations of motion for such systems. The approach follows Ref. 1. The method is applied to the Large Space Telescope.

Coordinate Systems and Assumptions

The considered model (Fig. 1) is described in terms of hybrid coordinates,[2] i.e., Kardan angles, for the deviation of the undeformed system from a given reference frame

$$q(t) = [\alpha(t),\ \ \beta(t),\ \ \gamma(t)]^T \tag{1}$$

(translational motion is not considered), and a deflection vector describing the elastic motion relative to the rigid body state

$$\bar{s}(y,t) = [\bar{u}(y,t),\ \ \bar{v}(y,t),\ \ \bar{w}(y,t)]^T \tag{2}$$

Both q and $\bar{s}$ describe the motion of the main body (Fig. 1), which will be of most interest. They are assumed to be small. The reference frame may rotate with a constant angular velocity $\mathbf{\Omega}$. Transverse shear as well as torsional influences are neglected.

Bending Oscillations

As the rigid body motion has no influence on geometrical boundary conditions, q and $\mathbf{\Omega}$ are set equal to zero in the first step. Then the partial differential equation governing the undamped plane motion of the main body is expressed using linear spatial differential operators[3]

$$\mathfrak{M}(\ddot{\bar{u}}) + \mathfrak{S}(\bar{u}) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\ \ ) = (\dot{\ \ }) \tag{3}$$

where

$$\mathfrak{M} = \rho A - \rho I_z \frac{\partial^2}{\partial y^2} \qquad \mathfrak{S} = EI_z \frac{\partial^4}{\partial y^4} \tag{4}$$

ρ describes the mass density, A is the cross-sectional area, I_z the polar momentum of inertia, and E Young's modulus. When the model is divided into sectors between the additional bodies, the boundary conditions for each segment are

$$y = y_k, \quad k = 0(1)p+1 : \mathfrak{R}0_0(\bar{u}) + \mathfrak{R}2_0(\ddot{\bar{u}}) = F_{Rk} \tag{5}$$

$$\mathfrak{R}0_1\left(\frac{\partial \bar{u}}{\partial y}\right) + \mathfrak{R}2_1\left(\frac{\partial \ddot{\bar{u}}}{\partial y}\right) = M_{Rk} \tag{6}$$

where

$$\mathfrak{R}0_0 = -EI_z \frac{\partial^3}{\partial y^3} \qquad \mathfrak{R}0_1 = EI_z \frac{\partial}{\partial y} \tag{7}$$

$$\mathfrak{R}2_0 = \rho I_z \frac{\partial}{\partial y} + m_k \qquad \mathfrak{R}2_1 = J_{zk} \tag{8}$$

Here, m_k, J_{zk} are the mass and moment of inertia of the kth additional body, p is the number of bodies, y_k describes the location of bodies and the boundaries of the unsegmented model ($k=0,\ p+1$); and F_R and M_R are reaction forces and torques, respectively. Usually, each sector is investigated separately. This can be avoided using the principle of virtual work

$$\delta W = \int_0^l \delta\bar{u}[\mathfrak{M}(\ddot{\bar{u}}) + \mathfrak{S}(\bar{u})]\mathrm{d}y + \sum_{k=0}^{p+1}\sum_{n=0}^{1} \delta\bar{u}^{(n)}[\mathfrak{R}0_n(\bar{u}^{(n)}) + \mathfrak{R}2_n(\ddot{\bar{u}}^{(n)})]\,|_{y_k} = 0 \tag{9}$$

with n as spatial derivative.

Introducing the eigenfunction expansion

$$\bar{u}(y,t) = \sum_{\infty} \bar{u}_i(y)\ \exp(\lambda_i t) = \bar{u}(y)^T g_u(t)\ ;\ \bar{u} = (\bar{u}_1 \cdots \bar{u}_\infty)^T \tag{10}$$

Equation (9) becomes time-independent. The resulting equation still will be satisfied if the virtual displacements are replaced by

$$\delta\bar{u} \to \bar{u}_j \tag{11}$$

since the eigenfunction $\bar{u}_j$ fulfills the time-independent part of Eqs. (3, 5, 6). This yields for the ith mode of Eq. (10)

$$\int_0^l \bar{u}_j[\lambda_i^2\mathfrak{M}(\bar{u}_i) + \mathfrak{S}(\bar{u}_i)]\mathrm{d}y + \sum_k\sum_n \bar{u}_j^{(n)}[\mathfrak{R}0_n(\bar{u}_i^{(n)}) + \lambda_i^2\,\mathfrak{R}2_n(\bar{u}_i^{(n)})]\,|_{y_k} = 0 \tag{12}$$

However, Eq. (12) may also be formulated for the jth mode of Eq. (10), when the virtual displacement is substituted for the

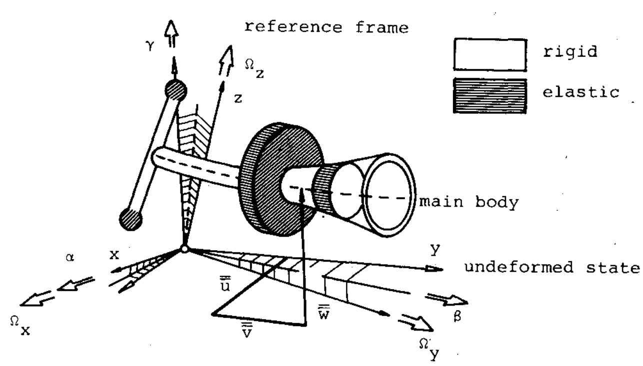

Fig. 1 Coordinate systems.

Received Aug. 25, 1977; synoptic received Jan. 24, 1978; revision received May 23, 1978.

Index category: Structural Dynamics.

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ith eigenfunction $\bar{u}_i$. When this is done and the difference between the two equations is formed, a generalized orthogonality condition is obtained

$$(\lambda_i^2-\lambda_j^2)\left[\int_0^l \bar{u}_j\mathfrak{M}(\bar{u}_i)\,\mathrm{d}y+\sum_k\sum_n \bar{u}_j^{(n)}\mathfrak{R}2_n(\bar{u}_i^{(n)})\,|_{y_k}\right]=0 \tag{13}$$

where $(\lambda_i^2-\lambda_j^2)\neq 0$ for $i\neq j$ (Ref. 3). For $i=j$, the second term in Eq. (13) may be normalized to unity. In this case, Eq. (12) can be resolved to

$$-\lambda_i^2=\omega_i^2=\left[\int_0^l \bar{u}_i\mathfrak{S}(u_i)\,\mathrm{d}y+\sum_k\sum_n \bar{u}_i^{(n)}\mathfrak{R}0_n(\bar{u}_i^{(n)})\,|_{y_k}\right]$$
$$\times\left[\int_0^l \bar{u}_i\mathfrak{M}(\bar{u}_i)\,\mathrm{d}y+\sum_k\sum_n \bar{u}_i^{(n)}\mathfrak{R}2_n(\bar{u}_i^{(n)})\,|_{y_k}\right]^{-1} \tag{14}$$

which is a generalized Rayleigh quotient. Equation (13) corresponds to a definition given in Ref. 5 for $n=0$.

Using a generalized Fourier expansion theorem

$$\bar{u}_i=\lim_{N\to\infty}(u^Ta_i)\qquad u=(u_1\cdots u_N)^T \tag{15}$$

Equation (13) becomes the quotient of two quadratic forms

$$\omega_i^2=[a_i^TK(u)a_i]\,[a_i^TM(u)a_i]^{-1} \tag{16}$$

where M and K are (generalized) mass and stiffness matrices, respectively. The stationarity condition of the Rayleigh quotient requires a differentiation with respect to the a_i yielding

$$(\omega_i^2M-K)a_i=0 \tag{17}$$

from which the eigenfunctions can be calculated.

System Equations

The approximation function u must only satisfy the geometrical boundary conditions. The kinetic ones are fulfilled by the generalized Rayleigh quotient [(Eq. (14)], and it is not necessary to calculate them explicitly. This leads to a modified use of Hamilton's principle for the general case **($q\neq 0$, $\Omega\neq 0$)**

$$\int_{t_0}^{t_1}\int_0^l\left\{L_{el}+\sum_{k=1}^{p}L_{rb}\delta(y-y_k)+\sum_{j=1}^{q}W\delta(y-y_j)\right\}\mathrm{d}y\mathrm{d}t$$
$$=\iint F\mathrm{d}y\mathrm{d}t\to\text{Extremum} \tag{18}$$

where $L=T-V$ is the Lagrange function; W the work done by external forces; δ the Dirac function; y_k, y_j location of additional bodies and forces, respectively; and q the number of forces. The Hamilton function F in Eq. (18) can be formulated as

$$F=F_{el}+F_{rb}+F_c \tag{19}$$

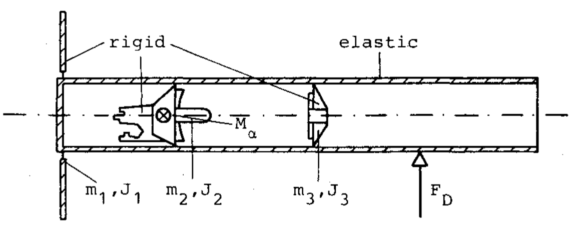


Fig. 2 Large space telescope (LST).

where F_{el}, F_{rb} are quadratic and linear in $(d^k/dt^k)(\partial^n\bar{s}/\partial y^n)$, $(d^k/dt^k)\ q$ respectively, and F_c is bilinear in both expressions. Using Eq. (11) $(j=1(1)\infty)$ and integrating with respect to time (not space), the variation of Eq. (18) yields

$$\sum_n\int_0^l\left\{\bar{u}^{(n)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_{el}}{\partial\dot{\bar{u}}^n}\right)-\left(\frac{\partial F_{el}}{\partial\bar{u}^n}\right)\right]\right.$$
$$\left.+\bar{u}^{(n)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_c}{\partial\dot{\bar{u}}^{(n)}}\right)-\left(\frac{\partial F_c}{\partial\bar{u}^n}\right)\right]\right\}\mathrm{d}y$$
$$+\sum_n\int_0^l\left\{\bar{v}^{(n)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_{el}}{\partial\dot{\bar{v}}^{(n)}}\right)-\left(\frac{\partial F_{el}}{\partial\bar{v}^{(n)}}\right)\right]\right.$$
$$\left.+\bar{v}^{(n)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_c}{\partial\dot{\bar{v}}^{(n)}}\right)-\left(\frac{\partial F_c}{\partial\bar{v}^{(n)}}\right)\right]\right\}\mathrm{d}y$$
$$+\sum_n\int_0^l\left\{\bar{w}^{(n)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_{el}}{\partial\dot{\bar{w}}^{(n)}}\right)-\left(\frac{\partial F_{el}}{\partial\bar{w}^{(n)}}\right)\right]\right.$$
$$\left.+\bar{w}^{(n)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_c}{\partial\dot{\bar{w}}^{(n)}}\right)-\left(\frac{\partial F_c}{\partial\bar{w}^{(n)}}\right)\right]\right\}\mathrm{d}y$$
$$+\delta q^T\sum_n\left\{\int_0^l\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_{rb}}{\partial\dot{q}}\right)-\left(\frac{\partial F_{rb}}{\partial q}\right)\right]^T\right.$$
$$\left.+\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_c}{\partial\dot{q}}\right)-\left(\frac{\partial F_c}{\partial q}\right)\right]^T\right\}\mathrm{d}y=0 \tag{20}$$

As $\bar{u}$, $\bar{v}$, $\bar{w}$, q_i are independent generalized coordinates, each of the four sums in Eq. (20) must be zero. The first one, corresponding to Eq. (12), and the third one describe lateral vibrations; the second one, containing longitudinal vibrations, can be dropped for $\|\bar{v}\|\ll|\bar{u},\bar{w}\|$. The fourth sum describes the rigid body motion. When the eigenfunctions from Eq. (17) are inserted, the orthogonality condition [Eq. (13)] holds. The resulting equations read $(x=[g_w^Tg_u^T]^T)$

$$\ddot{x}+(D_x+G_x)\dot{x}+K_xx+C_1\ddot{q}+C_2\dot{q}+C_3q=f_x \tag{21}$$

$$J\ddot{q}+(D_q+G_q)\dot{q}+K_qq+C_1^T\ddot{x}-C_2^T\dot{x}+C_3^Tx=f_q \tag{22}$$

Omitting the integration with respect to space in Eq. (18), the kinetic boundary conditions are implicitly fulfilled in Eqs. (21 and 22). Thus, Eq. (25) gives a general formalism to derive the equations of motion, using approximation functions which must satisfy geometrical boundary conditions only.

It can be shown that structural damping is pervasive through all coordinates for the present problem. Thus, $K_q<0$ shows that a rotation about the minimum axis of inertia leads to instability, and from $K_x<0$ it follows that centrifugal and

Fig. 3 LST eigenfunctions.

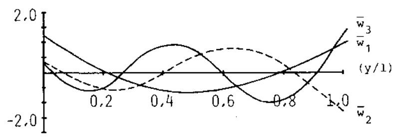
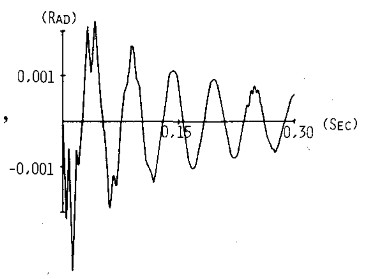


Fig. 4 Bending angle at m_2, uncontrolled system.



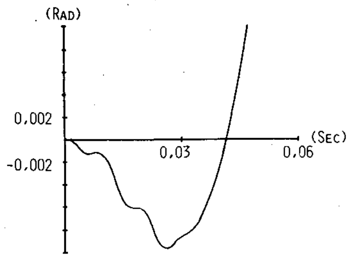


Fig. 5 Bending angle at m_2, unstable controller.

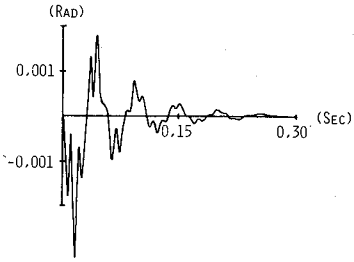


Fig. 6 Bending angle at m_2, optimized controller.

gyroscopic effects assigned to the elastic part of the system can destabilize the whole motion. On the other hand, positive restoring terms result in asymptotic stable behavior.

Application

The developed procedure is now applied to derive the equations of motion of the Large Space Telescope, which is a nonrotating flexible structure (Fig. 2). Since $\boldsymbol{\Omega}=\mathbf{0}$, C_2, and C_3 vanish, and for a plane motion in z direction, C_I reduces to a vector expression $\boldsymbol{c}$

$$c=\int_0^l \left\{\frac{y}{l}\bar{w}+\frac{1}{m}\rho I_x\frac{\partial \bar{w}}{\partial y}\right\}\mathrm{d}y+\sum_{i=1}^{3}\left\{\frac{m_i}{m}y\bar{w}+\frac{1}{m}J_{xi}\frac{\partial \bar{w}}{\partial y}\right\}\Big|_{y_i}, \quad (m=m_{el}) \tag{23}$$

The eigenfunctions $\bar{w}_i$ are plotted in Fig. 3; they are derived using 12 eigenfunctions of a free-free beam as approximation functions. Introducing modal damping, the equations of motion read

$$\ddot{x}+\mathrm{diag}(2\zeta\omega_i)\dot{x}+\mathrm{diag}(\omega_i^2)x=-c\ddot{\alpha}+\frac{\partial \bar{w}}{\partial y}\Big|_{y_M}M_\alpha+\bar{w}\,|_{y_F}F_D \tag{24}$$

$$(J/m)\ddot{\alpha}=-c^T\ddot{x}+M_\alpha+y_F F_D \tag{25}$$

where F_D is a disturbance force at $y=y_F$ and M_α a control torque at y_M. It is assumed, that the system is excited with an eccentric shock at $y_F=3/4\ l$. Figure 4 shows the uncontrolled system response (bending angle at m_2). If a control torque is considered which is time varying with a frequency near a bending frequency, the whole motion becomes unstable (Fig. 5). However, if M_α is optimized with respect to a quadratic cost functional using the well-known optimal regulator theory, then the rigid body motion as well as the bending oscillation can be kept small (Fig. 6).

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Eigensolution for Large Flexible Spacecraft with Singular Gyroscopic Matrices

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Introduction

THE advent of the large space structures (LSS) presents new problems in the dynamics and control areas. The large sizes envisioned and practical limits on lifting capacity require that the structural members be extremely light, which in turn implies a large degree of distributed structural flexibility and very low natural frequencies. No longer can the effects of the spacecraft flexibility be treated as a perturbation of the rigid body. In determining the system response to external excitation or to internal disturbances, the problem of mathematical modeling becomes critical, particularly if one wishes to obtain the response by modal analysis. The theory for the modal analysis of nonspinning systems with elastic restoring forces is well developed. The presence of stored angular momentum in nonspinning spacecraft, however, complicates the response problem because of the resulting gyroscopic effects. What complicates the response problem for a linear gyroscopic system is that the classical modal analysis will not uncouple the system equations of motion, so that the question of spacecraft modes requires a different interpretation. The question has been studied extensively by Meirovitch[1] for systems with a nonsingular gyroscopic matrix. A method was developed for determining the spacecraft modes, as well as a modal analysis for the evaluation of the response of gyroscopic systems.[2] This Note uses the free-free classical modes of the spacecraft (with all rotors locked) to reformulate the equations of motion using a reduced state vector. An extension must be made to Meirovitch's method[1] because of the singularity of the gyroscopic matrix involved in the present formulation. Subsequently, the problem is transformed into an eigenvalue problem for a single symmetric matrix.

Problem Formulation

Spinning rotors are employed as momentum exchange controllers to stabilize the spacecraft attitude, appearing in the form of reaction wheels, momentum wheels, or control moment gyros. Any combination of these can be present in a spacecraft and they can be distributed over the spacecraft structure. Our interest lies in constant relative speed of the rotor and constant relative orientation of its spin axis with respect to the mounting location. The spacecraft structure is assumed to be stabilized relative to an inertial space, so that

Presented as Paper 77-1096 at the AIAA 1977 Guidance and Control Conference, Hollywood, Fla., Aug. 8-10, 1977; submitted Oct. 7, 1977; revision received July 11, 1978.

Editor's Note: Dr. Canavin died in the September 25 air crash in San Diego.

Index categories: Spacecraft Dynamics and Control; Structural Dynamics.

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